APPROXIMATE ANALYSIS OF THE STEADY-STATE TEMPERATURE FIELD OF A PARALLELEPIPED WITH A LOCAL ENERGY SOURCE

G. N. Dul'nev and É. I. Ermolina

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There is an approximate analysis of the steady-state temperature field of a parallelepiped on one of whose faces there is a local energy source. A practical calculation scheme is proposed, and its accuracy is evaluated.

We consider a parallelepiped on whose upper face there is an energy source of rectangular shape; the fact itself does not dissipate energy in the surrounding medium. At the opposite face there are boundary conditions of the third kind; the lateral faces of the parallelepiped are insulated (Fig. 1). Problems of this type arise in many applications, primarily in microelectronics. There are no fundamental difficulties involved in analyzing the temperature field of such an object, but the complexity of the resulting equations makes them difficult to use in practice, so that it is necessary to use instead the comparatively simple calculation methods worked out for particular cases [1, 2]. Accordingly, there is a need for an approximate solution method which would provide the necessary accuracy in the first approximation and whose equations would be comparatively simple. One such method is the so-called generalized Kantorovich method, whose basis is given in [3, 4]. Below we use this method to solve the problem formulated above, and we point out a way to simplify the calculation equations without reducing their accuracy.

The mathematical formulation of the problem for the case in which the source is the central position reduces to the solution of the differential equation

$$\varepsilon_x \frac{\partial^2 N}{\partial X^2} + \varepsilon_y \frac{\partial^2 N}{\partial Y^2} + \frac{\partial^2 N}{\partial Z^2} = 0 \tag{1}$$

with the boundary conditions

$$\left.\frac{\partial N}{\partial X}\right|_{X=0} = 0; \quad \left.\frac{\partial N}{\partial Y}\right|_{Y=0} = 0; \quad \left.\frac{\partial N}{\partial X}\right|_{X=1} = 0; \quad \left.\frac{\partial N}{\partial Y}\right|_{Y=1} = 0;$$

$$\left.\left[\frac{\partial N}{\partial Z} + \mathrm{Bi}\, N\right]\right|_{Z=1} = 0; \quad \left.\frac{\partial N}{\partial Z}\right|_{Z=0} = 1\{\delta_x\}\, 1\{\delta_y\}; \tag{2}$$

$$1\{\delta_i\} = \begin{cases} 1, & |i| \leqslant \delta_i/2, \\ 0, & |i| > \delta_i/2, \end{cases} \quad i = X,\ Y.$$

Here we have used the dimensionless parameters

$$N = \frac{\vartheta\lambda}{q_s L_3}; \quad X = \frac{2x}{L_1}; \quad Y = \frac{2y}{L_2}; \quad Z = \frac{z}{L_3};$$
$$\varepsilon_x = \left(\frac{2L_3}{L_1}\right)^2; \quad \varepsilon_y = \left(\frac{2L_3}{L_2}\right)^2; \quad \mathrm{Bi} = \frac{\alpha L_3}{\lambda}; \quad \delta_x = \frac{l_1}{L_1}; \quad \delta_y = \frac{l_2}{L_2}. \tag{3}$$

According to the generalized Kantorovich method this problem is equivalent to that of minimizing the functional [4]

$$J[N] = \int_0^1\int_0^1\int_0^1 \left\{ \varepsilon_x \left(\frac{\partial N}{\partial X}\right)^2 + \varepsilon_y \left(\frac{\partial N}{\partial Y}\right)^2 + \left(\frac{\partial N}{\partial Z}\right)^2 \right\} dY dZ + \int_0^1\int_0^1 \mathrm{Bi}\, N^2_{|Z=1}\, dX dY + 2\int_0^1\int_0^1 1\{\delta_x\}\, 1\{\delta_y\}\, N_{|Z=0}\, dX dY, \tag{4}$$

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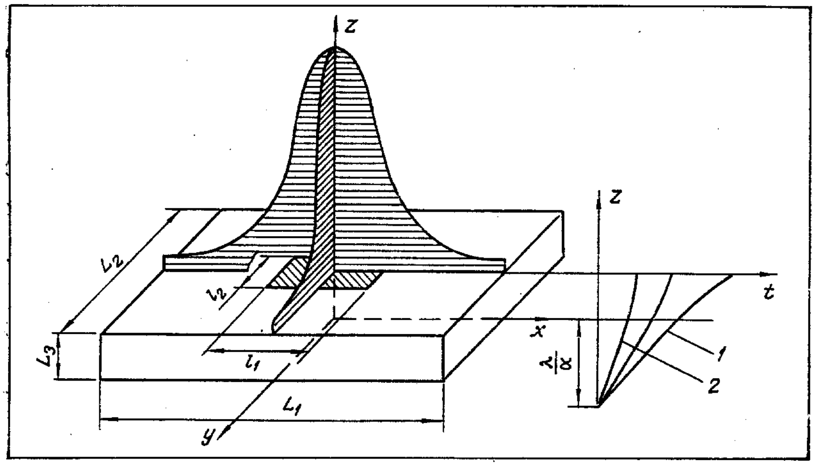

Fig. 1. Parallelepiped with a local source. 1) x, y $\in [l_1 \times l_2]$; 2) x, y $\overline{\in} [l_1 \times l_2]$.

where the unknown function N can be approximated by

$$N \approx N(X)\, M(Y)\, Q(Z), \tag{5}$$

and the coordinate dependences N(X), M(γ), and Q(Z) are determined by the averaging method of [5].

With the same goal in mind, we subject Eq. (1) and boundary conditions (2) to the averaging operator

$$I_{YZ}[N] = \int_0^1\int_0^1 N\,dY\,dZ = \langle N^{I}_{YZ}(X)\rangle, \tag{6}$$

Finding an ordinary differential equation for $\langle N^{I}_{YZ}(X)\rangle$:

$$\frac{d^2\langle N^{I}_{YZ}(X)\rangle}{dX^2} - (p^{I}_x)^2\,\langle N^{I}_{YZ}(X)\rangle = -\frac{\delta_y}{\varepsilon_x}\,1\{\delta_x\}. \tag{7}$$

Here the boundary conditions are

$$\left.\frac{d\,\langle N^{I}_{YZ}(X)\rangle}{dX}\right|_{X=0} = 0; \quad \left.\frac{d\,\langle N_{YZ}(X)\rangle}{dX}\right|_{X=1} = 0, \tag{8}$$

and we have adopted the notation

$$\varepsilon_x (p^{I}_x)^2 = \mathrm{Bi}\,\psi_z, \tag{9}$$

$$\psi_z = \frac{\int_0^1 N(X, Y, Z)_{|Z=1}\,dY}{\langle N^{I}_{YZ}(X)\rangle} \cong \frac{\int_0^1\int_0^1 N(X, Y, Z)_{|Z=1}\,dX\,dY}{\int_0^1\int_0^1\int_0^1 N(X, Y, Z)\,dX\,dY\,dZ}. \tag{10}$$

The coefficient ψ_Z is a measure of the deviation from a uniform temperature field along the z axis. Under boundary conditions (8), the solution of Eq. (7) is

$$\langle N^{I}_{YZ}(X)\rangle = \frac{\delta_y}{\varepsilon_x (p^{I}_x)^2}\,\varphi(X), \tag{11}$$

$$\varphi(X) = \begin{cases} 1 - \dfrac{\operatorname{sh} p^{I}_x (1-\delta_x)}{\operatorname{sh} p^{I}_x}\operatorname{ch} p^{I}_x X, & |X| \leqslant \delta_x/2, \\ \dfrac{\operatorname{sh} p^{I}_x \delta_x}{\operatorname{sh} p^{I}_x}\operatorname{ch} p^{I}_x (1-X), & |X| > \delta_x/2. \end{cases} \tag{12}$$

Setting $N(X) \approx \langle N^{I}_{YZ}(X)\rangle$ in (5) and substituting the result found for N into (4), we find [5]

$$J[S(Y, Z)] = \int_0^1\int_0^1 \left\{a_1\left[\varepsilon_y\left(\frac{\partial S}{\partial Y}\right)^2 + \left(\frac{\partial S}{\partial Z}\right)^2\right] + a_2 S^2\right\} dY\,dZ + a_1\,\mathrm{Bi}\int_0^1 S^2|_{Z=1}\,dY + 2a_3\int_0^1 1\{\delta_y\}\,S|_{Z=0}\,dY, \tag{13}$$

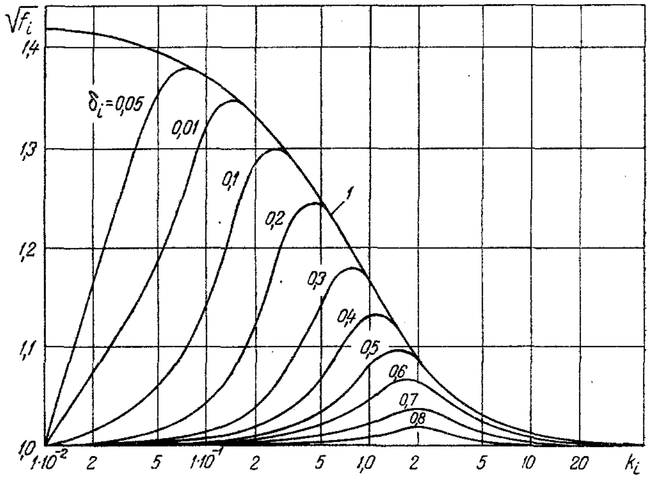

Fig. 2. The function $\sqrt{f_i} = \varphi(\delta_i, k_i)$.

$$S(Y, Z) = M(Y)\,Q(Z); \tag{14}$$

$$a_1 = \int_0^1 [\langle N_{YZ}^{I}(X)\rangle]^2 dX = \left[\frac{\delta_y}{\varepsilon_x (p_x^{I})^2}\right]^2 \Phi_{1x}^{I},$$

$$a_2 = \varepsilon_x \int_0^1 \left\{\frac{d}{dX}\langle N_{YZ}^{I}(X)\rangle\right\}^2 dX = \frac{\delta_y}{\varepsilon_x (p_x^{I})^2}\Phi_{2x}^{I}, \tag{15}$$

$$a_3 = \int_0^1 1\{\delta_x\}\,\langle N_{YZ}^{I}(X)\rangle\, dX = \frac{\delta_y}{\varepsilon_x (p_x^{I})^2}\Phi_{3x}^{I}.$$

Here S(Y, Z) is the function which minimizes functional (13) and is the solution of the boundary-value problem

$$\varepsilon_y \frac{\partial^2 S}{\partial Y^2} + \frac{\partial^2 S}{\partial Z^2} - \frac{a_2}{a_1} S = 0, \tag{16}$$

$$\left.\frac{\partial S}{\partial Y}\right|_{Y=0} = 0; \quad \left.\frac{\partial S}{\partial Y}\right|_{Y=1} = 0,$$

$$\left.\left[\frac{\partial S}{\partial Z} + \mathrm{Bi}\,S\right]\right|_{Z=1} = 0; \quad \left.\frac{\partial S}{\partial Z}\right|_{Z=0} = -\frac{a_3}{a_1}\,1\{\delta_x\}. \tag{17}$$

Accordingly, the original boundary-value problem in (1), (2), reduces to the two-dimensional problem in (16), (17), to solve which we again use the averaging method.

The application of operator

$$\mathrm{I}_Z[S] = \int_0^1 S(Y, Z)\,dZ = \langle M_Z^{II}(Y)\rangle \tag{18}$$

to Eq. (16) and the boundary conditions (17) yields

$$\frac{d^2\langle M_Z^{II}(Y)\rangle}{dY^2} - (p_y^{II})^2\langle M_Z^{II}(Y)\rangle = -\frac{a_3}{a_1}\cdot\frac{1}{\varepsilon_y}\,1\{\delta_y\}, \tag{19}$$

$$\left.\frac{d\langle M_Z^{II}(Y)\rangle}{dY}\right|_{Y=0} = 0; \quad \left.\frac{d\langle M_Z^{II}(Y)\rangle}{dY}\right|_{Y=1} = 0. \tag{20}$$

A solution of Eq. (19) compatible with boundary conditions (20) is

$$\langle M_Z^{II}(Y)\rangle = \frac{a_3}{a_1}\frac{1}{\varepsilon_y (p_y^{II})^2}\varphi(Y), \tag{21}$$

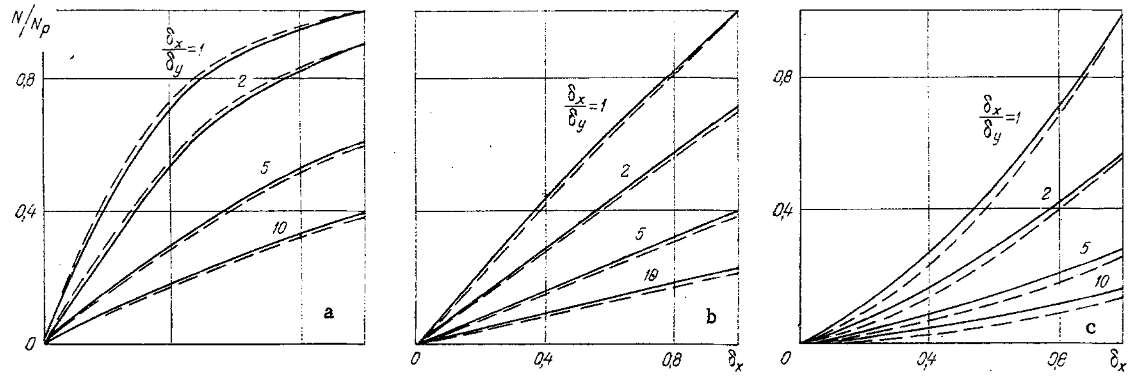

Fig. 3. Comparison of the exact (solid curves) and approximate (dashed curves) methods of solving the problem for the case $L_2/L_1 = 1$ and for values of δ_x/δ_y from 1 to 10. Bi = 0.2. a) $L_3/L_1 = 0.05$; b) $L_3/L_1 = 0.1$; c) $L_3/L_1 = 0.2$.

$$\varphi(Y) = \begin{cases} 1 - \dfrac{\operatorname{sh} p_y^{\mathrm{II}}(1-\delta_y)}{\operatorname{sh} p_y^{\mathrm{II}}} \operatorname{ch} p_y^{\mathrm{II}} Y, & |Y| \leqslant \delta_y/2, \\ \dfrac{\operatorname{sh} p_y^{\mathrm{II}} \delta_y}{\operatorname{sh} p_y^{\mathrm{II}}} \operatorname{ch} p_y^{\mathrm{II}}(1-Y), & |Y| > \delta_y/2, \end{cases} \tag{22}$$

$$\varepsilon_y (p_y^{\mathrm{II}})^2 = \mathrm{Bi}\,\psi_z + \frac{a_2}{a_1} = \mathrm{Bi}\,\psi_z \left(1 + \frac{\Phi_{2x}^{\mathrm{I}}}{\Phi_{1x}^{\mathrm{I}}}\right). \tag{23}$$

Accordingly, the approximate solution of this problem is

$$N(X, Y, Z) = \langle N_{YZ}^{\mathrm{I}}(X) \rangle \ \langle M_Z^{\mathrm{II}}(Y) \rangle \ Q(Z). \tag{24}$$

If we use a different averaging order [if we first apply the operator I_{XZ} to Eq. (1) and then apply the operator I_Z to the resulting two-dimensional differential equation], the solution turns out to be analogous:

$$N(X, Y, Z) = \langle N_Z^{\mathrm{II}}(X) \rangle \ \langle M_{XZ}^{\mathrm{I}}(Y) \rangle \ Q(Z). \tag{25}$$

Equations (24) and (25) differ only in the structure of the parameters p_x and p_y. Analysis shows that the parameter p corresponding to the coordinate function determined second [p_y^{II} for Eq. (24) and p_x^{II} for Eq. (25)] gives the temperature dependence in this direction more accurately. We can thus write the unknown function as

$$N = \langle N^{\mathrm{II}}(X) \rangle \ \langle M^{\mathrm{II}}(Y) \rangle \ Q(Z). \tag{26}$$

To determine the function Q(Z) we substitute (26) into (4) and integrate the latter over X and Y; in this manner we reduce the problem to one of finding the minimum of a simple integral:

$$J[Q(Z)] = \int_0^1 b_{1x} b_{1y} \left(\frac{dQ(Z)}{dZ}\right)^2 dZ + \int_0^1 (b_{1x} b_{2y} + b_{1y} b_{2x}) Q^2(Z)\, dZ + 2 b_{3x} b_{3y} Q(Z)_{|Z=0} + b_{1x} b_{1y} \mathrm{Bi}\, Q^2(Z)_{|Z=1}, \tag{27}$$

$$b_{1i} = \int_0^1 [\langle R_Z(i) \rangle]^2 di = \frac{1}{\delta_i^2} \Phi_{1i}^{\mathrm{II}}, \tag{28}$$

$$b_{2i} = \varepsilon_i \int_0^1 \left\{\frac{d}{di} [\langle R_Z(i) \rangle]\right\}^2 di = \frac{\varepsilon_i (p_i^{\mathrm{II}})^2}{\delta_i^2} \Phi_{2i}^{\mathrm{II}}, \tag{29}$$

$$b_{3i} = \int_0^1 1\{\delta_i\} \langle R_Z(i) \rangle\, di = \frac{1}{\delta_i} \Phi_{3i}^{\mathrm{II}}, \tag{30}$$

$$i \in [X, Y], \ \langle R_Z(X) \rangle = \langle N_Z^{\mathrm{II}}(X) \rangle, \ \langle R_Z(Y) \rangle = \langle M_Z^{\mathrm{II}}(Y) \rangle.$$

The function Q(Z) which minimizes functional (27) must satisfy the equation

$$\frac{d^2Q(Z)}{dZ^2} - p_z^2 Q(Z) = 0 \tag{31}$$

and the boundary conditions

$$\left[\frac{dQ(Z)}{dZ} + \mathrm{Bi}\, Q(Z)\right]\Bigg|_{Z=1} = 0; \quad \frac{dQ(Z)}{dZ}\Bigg|_{Z=0} = \frac{b_{3x}b_{3y}}{b_{1x}b_{1y}}; \tag{32}$$

$$p_z^2 = \frac{b_{2x}}{b_{1x}} + \frac{b_{2y}}{b_{1y}}. \tag{33}$$

The solution of this equation is

$$Q(Z) = \frac{1}{p_z}\frac{b_{3x}b_{3y}}{b_{1x}b_{1y}}\varphi(Z), \tag{34}$$

$$\varphi(Z) = \left(\frac{p_z/\mathrm{Bi} + \mathrm{th}\, p_z}{1 + \frac{p_z}{\mathrm{Bi}}\,\mathrm{th}\, p_z} - \mathrm{th}\, p_z Z\right)\mathrm{ch}\, p_z Z. \tag{35}$$

Substituting the expressions for $\langle N_Z^{II}(X)\rangle$, $\langle M_Z^{II}(Y)\rangle$, and Q(Z) into (26), and using (28)-(30), we find

$$N = \frac{\Phi_{3x}^{II}\Phi_{3y}^{II}}{\Phi_{1x}^{II}\Phi_{1y}^{II}}\frac{1}{p_z}\varphi(X)\varphi(Y)\varphi(Z). \tag{36}$$

In turn, we have

$$\Phi_{3i}^{II} = \delta_i - \frac{\mathrm{sh}\, p_i^{II}(1-\delta_i)}{p_i^{II}\,\mathrm{sh}\, p_i^{II}}\mathrm{sh}\, p_i^{II}\delta_i, \tag{37}$$

$$\Phi_{1i}^{II} = \frac{1}{2}\left\{3\Phi_{3i}^{II} - \delta_i + \delta_i\frac{\mathrm{sh}\, p_i^{II}(1-2\delta_i)}{\mathrm{sh}\, p_i^{II}} + \frac{\mathrm{sh}^2\, p_i^{II}\delta_i}{\mathrm{sh}^2\, p_i^{II}}\right\}, \tag{38}$$

$$\Phi_{2i}^{II} = \Phi_{3i}^{II} - \Phi_{1i}^{II}, \tag{39}$$

and the expressions for p_x^{II}, p_y^{II}, p_z, and ψ_z, are according to Eqs. (23), (33), (28)-(30), (9), and (10),

$$\varepsilon_x (p_x^{II})^2 = \mathrm{Bi}\,\psi_z f_y^I, \tag{40}$$

$$\varepsilon_y (p_y^{II})^2 = \mathrm{Bi}\,\psi_z f_x^I, \tag{41}$$

$$p_z^2 = \varepsilon_x (p_x^{II})^2 (f_x^{II} - 1) + \varepsilon_y (p_y^{II})^2 (f_y^{II} - 1), \tag{42}$$

$$\psi_z = \frac{\varphi(Z)_{|Z=1}}{\int_0^1 \varphi(Z)\,dZ} = \frac{p_z^2}{\mathrm{Bi}}\cdot\frac{1}{\mathrm{ch}\, p_z + \frac{p_z}{\mathrm{Bi}}\,\mathrm{sh}\, p_z - 1}, \tag{43}$$

where

$$f_i^{II} = \frac{\Phi_{3i}^{II}}{\Phi_{1i}^{II}}; \quad f_i^I = \frac{\Phi_{3i}^I}{\Phi_{1i}^I}. \tag{44}$$

The analytic expressions for Φ_{3i}^I and Φ_{1i}^I are like those for Φ_{3i}^{II} and Φ_{1i}^{II} with p_i^{II} replaced by p_i^I.

Accordingly, we have found an expression for the dimensionless temperature N and equations for all the parameters which appear in this expression. However, it is quite laborious to carry out calculations on the basis of these equations. The primary difficulty lies in the determination of the parameters Φ_i and the need to solve the system of transcendental Eqs. (40)-(43), which relate the parameters p_x, p_y, p_z and ψ_z. A simple iterative method is usually used to determine them.

To improve the accuracy of these equations and to simplify them we take the following approach We define the complex $f_x^{II} f_y^{II}$ in such a manner that the resulting solution satisfies the heat-balance equation at the surface $z = L_3$:

$$\alpha L_1 L_2 \bar{\vartheta}_{|z=L_3} = P; \quad \bar{\vartheta}_{|z=L_3} = \frac{\bar{N}_{|Z=1} q_s L_3}{\lambda}; \quad \bar{N}_{|Z=1} = \int_0^1\int_0^1 N_{|Z=1}\,dX\,dY.$$

Carrying out the integration and the necessary algebraic manipulations we find

$$f_x^{II} f_y^{II} = \operatorname{ch} p_z + \frac{p_z}{\mathrm{Bi}} \operatorname{sh} p_z,$$

and thus

$$N = \left(\frac{\operatorname{ch} p_z}{p_z} + \frac{\operatorname{sh} p_z}{\mathrm{Bi}}\right) \varphi(X)\, \varphi(Y)\, \varphi(Z). \tag{45}$$

Analysis of f_i as a function of its parameters shows that the quantity f_i^{II} differs little from f_i^{I}; in the limit $\delta_i \to 1$ we have $f_i = 1$, while in the limit $\delta_i \to 0$ we have

$$\lim_{\delta_i \to 0} f_i = 2\,\frac{2k_i - 1 + \exp(-2k_i)}{4k_i - 3 + (2k_i - 3)\exp(-2k_i)},$$

where

$$k_i = \frac{\sqrt{\mathrm{Bi}\,\psi_z}}{\Delta_i}; \quad \Delta_i = \frac{2L_3}{l_i}. \tag{46}$$

In turn, we have

$$\lim_{\substack{\delta_i \to 0 \\ \Delta_i \to 0}} f_i = 2.$$

Figure 2 shows $\sqrt{f_i}$ as a function of the parameters k_i and δ_i. Curve 1 is found from (46). All the curves, corresponding to various values of δ_i, gradually approach curve 1. This diagram greatly simplifies the calculation of the parameters which appear in the expression for N.

Analysis shows that the value of $\psi_z^{(n)}$ obtained after the successive approximations is approximately the same as the initially specified value $\psi_z^{(1)}$:

$$\psi_z^{(n)} \approx \psi_z^{(1)} = \frac{2}{2 + \mathrm{Bi}}. \tag{47}$$

Let us outline the order for calculating the dimensionless temperature N.

1. From Eqs. (3) we calculate δ_i, X, Y, and Z; from Eqs. (46) and (47) we calculate Δ_i, k_i, ψ_z.
2. From Fig. 2 we determine the values of $\sqrt{f_i}$ for the calculated values of k_i and δ_i.
3. From Eqs. (40)-(42) we determine p_x, p_y, p_z.
4. From Eqs. (12) and (22) we calculate the coordinate functions $\varphi(X)$ and $\varphi(Y)$; from (35) we calculate $\varphi(Z)$.
5. From Eq. (45) we determine the value of the dimensionless temperature N at the point under consideration.

The results of calculations of the dimensionless temperature N for the center of the source by this procedure for the case Bi = 0.2 and $L_1/L_2 = 1$ are shown by the dashed curves in Fig. 3a, b, and c. Also shown in this figure (solid curves) are the results calculated by the exact method [1].

The quantity plotted along the ordinate is N/N_p, where N_p is the dimensionless temperature for a source with dimensions equal to those of the base of the parallelepiped.

Comparison of the calculated results shows that with $L_3/L_1 \le 0.1$ and $\delta_i \ge 0.1$ the discrepancy between the results does not exceed 10%. At $L_3/L_1 > 0.1$ and $\delta_i < 0.6$ the discrepancies are larger, reaching 50% at $\delta_i < 0.2$. The reason for these large discrepancies is the pronounced nonuniformity of the temperature field for parallelepipeds with sides related in this manner.

NOTATION

l_1, l_2	are the dimensions of the source;
L_1, L_2, L_3	are the dimensions of the parallelepiped;
λ	is the thermal conductivity;
α	is the heat-transfer coefficient;
P	is the power of energy source;
t, t_c	are the temperatures of the object and the surrounding medium;

$\vartheta = t - t_c$ is the superheating;
$N = \vartheta\lambda / q_S L_3$ is the dimensionless temperature;
$q_S = P / l_1 l_2$ is the heat flux.

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